Combinatorial Games

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Abstract

We begin by developing the standard theory of normal play impartial games. Then misère quotients are introduced, which we will use to prove a powerful periodicity theorem for both normal and misère octal games. Finally, we develop the nim product to form the field of nimbers and solve a related computationally interesting game using the tartan theorem.

1 Introduction

We are concerned with deterministic, turn based two player games with perfect information. The analysis of such games is a rich and complex field, blooming with beautiful theories. In this article, we will restrict our attention to impartial games - those games in which both players have access to the same set of moves in any given position.

Games in which the player who makes the final move wins are called normal play games. The general theory of normal play impartial games is simple; each position is equivalent to another position from the completely understood game of nim. So if we can determine each position's associated nim position, we will have solved our game. Less understood is the theory of misère play impartial games, in which the last player to move loses. Whilst the elegant theory crumbles away, for many games, we can recover almost everything with an algebraic object known as the misère quotient. We will introduce this construction and use it to prove a general criterion which completely solves a large class of games. Even in normal play, many impartial games do not yield to these powerful correspondences and methods of combinatorial interest are required. We will finish by solving one such game using an elegant theorem which harnesses a field structure on the associated nim positions of normal play impartial games.

My introduction to this area is through mathematical programming problems; the theory of impartial games is particularly conducive to an algorithmic approach - with many recursively defined games and nicer still, simple theories equipping games with computationally efficient operations. This aspect is explored occasionally throughout, but the main focus is a rigorous exposition.

2 The general theory of normal play impartial games

In this section, we adopt the "normal play convention"; a player who cannot move loses.

We aim to devise a metric to measure the strength of a position. If possible we would like a metric which interacts nicely with the recursive structure often present in games; a position is defined recursively in terms of its *options*. The set of options of a position in a game is the set of positions reachable in one move from that position. In a game with no drawn positions, each position must either be winning for the previous player (a $\mathcal{P}$-position) or winning for the next player (an $\mathcal{N}$-position). A position which is winning for the previous player must have all its options leading to positions which are again winning for this player - all the options of $\mathcal{P}$-positions are $\mathcal{N}$-positions. Similarly an $\mathcal{N}$-position must have at least one option which is a $\mathcal{P}$-position. This already allows for winning positions to be found recursively. Labelling the terminal positions, called *endgames*, as $\mathcal{P}$-positions, using the previously mentioned rules, the outcome class of all other positions can be deduced. This algorithm is effective for providing a database to solve small games but projecting the information of a position into a binary parameter is far too restrictive to obtain a rich theory.

We will instead evaluate each position as a number. For more general games, these can be ordinal numbers, elements of a commutative monoid (as we shall see) or even special numbers with curious properties [Con][Elw]. But for finite (there can be no infinitely long sequence of options) impartial games with normal play, we shall see that the non-negative integers suffice.

2.1 Nim heaps

Take the ancient game of nim, played with piles of stones. In this game, the players alternate taking turns to select a pile and remove some number of stones from it. The player who is left with no stones to remove is the loser. We can view a position P of a game as the set of all move options available, so a single heap of stones in nim (called a nim heap) can be represented recursively as $*n = \{*n-1, *n-2, \cdots, *0\}$, where $*n$ denotes a heap of n stones and $*0 = \emptyset$ as there are no legal moves acting on a pile with no stones.

2.2 Smith's theory

The following construction due to Smith [Con] reveals the pleasant structure of normal play impartial games - they are simply nim heaps in disguise! We adopt an informal yet rigorous approach.

Consider the graph of a game with finitely many possible positions - with vertices as positions and directed edges as legal moves. Each vertex is initially assigned a weight of ∞ - such vertices will be called "unmarked". Begin by marking all positions with out-degree 0 with the label 0. These are $\mathcal{P}$-positions, as the player stuck in this position has no legal moves and therefore loses by the normal play convention. We will proceed to mark the vertex P with the value n iff both

(1) n is the smallest positive integer not assigned to a neighbouring vertex.

(2) Each option of P with mark $> n$ (i.e adjacent vertices of weight $> n$) must have an option marked as n.

Now assign values to each vertex according to the rules until it is no longer possible to mark any further positions. This algorithm will mark all the $\mathcal{P}$-positions with the label 0 and $\mathcal{N}$-positions with a positive integer.

The following paragraph only applies to games with infinite sequences of options and can be skipped. Then adorn each unmarked edge with the values of all neighbouring marked vertices as in $\infty_{a,b,c,\cdots}$. By condition (2), every unmarked position must neighbour an unmarked position which is not adorned by any extra values. Such a position is considered drawn by infinite repetition as a player can force the game to remain at unmarked positions. Marked positions with adorned values can be rescued by the first player by moving to one of the adjacent marked positions, otherwise they can force a draw by infinite repetition. From now on however, we will not be concerned with games which admit drawn positions.

We now claim that a position P with mark n has an equivalent structure to the nim heap $*n$. Suppose that this is the case for the first k positions labelled. Now consider the position labelled n at time $k+1$. Indeed, by condition (1), the adjacent nodes contain positions with any given mark r smaller than n, corresponding to the the position $*r$. Suppose instead the next player visits a position with a greater mark, then by condition (2) the previous player can visit a new position Q with value n once again. There must exist some Q labelled before P with this value - otherwise P would never have been marked as n. By assumption, Q is the position $*n$. Since only finitely many nodes have been marked, there must be finitely many nodes with value $> n$, so after finitely many waiting moves, the play can always return to a position marked n with no moves of larger size. Therefore it is of no strategic gain for the second player to utilise these waiting moves and we can indeed consider a position with mark n to be structurally equivalent to $*n = \{*n-1, *n-2, \cdots, *0\}$. We call the mark of a position its *nimber* or *nim value.*

This proves the famous theorem attributed to R.P Sprague and P.M Grundy.

Theorem 1. (Sprague-Grundy Theorem) *Every finite impartial game under the normal play convention is structurally equivalent to a single nim heap. A function taking positions to nim values is called a* Sprague-Grundy function*.*

Corollary: But the previous construction reveals more: the nimber of a position can be obtained by recursively applying (1).

2.3 Sums of games

We now apply the corollary of Smith's construction to recreate the theory of combining normal play impartial games. The *disjunctive sum* of a set of subgames is a new game such that on every move the next player must choose one subgame in which to play a single move. Many games can be written naturally in terms of a disjunctive sum of smaller games, and so a way of evaluating the positions of a disjunctive sum of games in terms of its subgames would be highly desirable. This is what we now develop.

For convenience, we define $\mathbf{N}=\{0,1,2,\ldots\}$. Also define the *minimal excludant* of a set $X \subset \mathbf{N}$ as $mex(X) := min(\mathbf{N} - X)$ to be the least non-negative integer not in X. According to (1), the nimber of a position is the *mex* of all the position's options.

By the Sprague-Grundy theorem, a sum of normal play impartial games must be equivalent to a sum of nim heaps, which is in turn equivalent to a single nim heap. So in order to study this class of games, it is sufficient to develop a suitable theory of nim.

2.3.1 The nim sum

Consider a game of nim with two heaps, one with a stones and the other with b stones. Define the nimber of this game to be $a \oplus b$, called the *nim sum* of a and b. A move in this game is to decrease the number of stones in one of the piles. From (1), it follows that $a \oplus b = mex(\{a' \oplus b | a' < a\} \cup \{a \oplus b' | b' < b\})$; this is just the mex of all the nim values of the options. Of course $0 \oplus 0 = 0$ as all $\mathcal{P}$-positions have nim value 0.

Lemma 2 *$\oplus$ is associative.*

Proof. Assume that $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ when $a+b+c \leq n$. So $(a \oplus b) \oplus (c+1)$

$= mex(\{(a' \oplus b) \oplus (c+1) | a' < a\} \cup \{(a \oplus b') \oplus (c+1) | b' < b\} \cup \{(a \oplus b) \oplus c' | c' \leq c\})$

$= mex(\{a' \oplus (b \oplus (c+1)) | a' < a\} \cup \{a \oplus (b' \oplus (c+1)) | b' < b\} \cup \{(a \oplus (b \oplus c') | c' \leq c\})$

$= a \oplus (b \oplus (c+1))$

An identical argument holds for the two other components. For the base case, note that $1 \oplus 0 = 1 = 0 \oplus 1$, so the lemma follows by induction.

By playing some small games of nim, we can deduce properties of $\oplus$. Observe the simple yet important identity $x \oplus x = 0$. Indeed, the game of nim with two equal heaps is a win for the second player, as they may simply copy the move of their opponent but in the opposite heap. So, employing the Sprague-Grundy theorem, this game is equivalent to a single nim heap of size 0. Furthermore, $\oplus$ is commutative as the heaps are unordered. Finally, adding a new heap of zero stones does not alter the game, so $x \oplus 0 = x$.

Corollary 3 *$(\mathbf{N}, \oplus)$ is an abelian group with identity 0 and $x^{-1} = x$.*

Proposition 4 *$a \oplus b \leq a + b$.*

Proof. Assume true for $a + b \leq n$. Then

$$a \oplus (b+1) = mex(\{a' \oplus (b+1) | a' < a\} \cup \{a \oplus b' | b' \leq b\})$$

By assumption, each option has nimber $\leq n$, so indeed $a \oplus (b+1) \leq n+1$. The claim follows by induction and commutativity of $\oplus$.

Theorem 5 *$\oplus$ is the bitwise* XOR *operator.*

Proof. For fixed $n \geq 1$, consider the subgroup of $(\mathbf{N}, \oplus)$ generated by $G_n := \langle 2^0, 2^1, \ldots, 2^n \rangle$. Each generator represents a different game of nim, so they are unique. We prove that the generators are independent by induction. Indeed by proposition 4

$$2^{k+1} > 2^{k+1} - 1 = \sum_{i=0}^{k} 2^i \geq \sum_{i \in I} 2^i \geq \bigoplus_{i \in I} 2^i$$

for every $I \subset \{1, 2, \ldots, k\}$. Therefore, 2^{k+1} is not contained in G_k and we conclude that the listed generators of G_n are independent.

G_n contains at least n distinct subgroups of order 2 of the form $\langle 0, 2^k \rangle$. Since it is abelian, it also contains a copy of $(\mathbf{Z}/2)^n$ as a subgroup. This has 2^n elements which is the also the maximum possible number of elements in G_n, so we conclude that $G_n \cong (\mathbf{Z}/2)^n$. By independence of the generators, note that $2^a \oplus 2^b = 2^a + 2^b$. Therefore, by taking n large enough and writing each nimber in binary, we conclude that the nim sum is identical to the bitwise XOR operator when acting on two finite nim heaps.

Written succinctly, where $+$ denotes the disjunctive sum of games and $\mathcal{G}(G)$ denotes the nim value of the position G, the Sprague-Grundy theorem says:

$$(*) \quad \mathcal{G}(X_1 + \ldots + X_2) = \mathcal{G}(X_1) \oplus \ldots \oplus \mathcal{G}(X_n)$$

Normal play impartial games are the subject of many mathematical programming problems. This is probably as a result of the simple theory, attracting a wider audience of problem solvers - where problems find their difficulty more in the creativity of the solver. Theorem 5 explains the volume of programming problems in this direction, as once a game has been dissolved into a question about binary bitwise operations and recursively generated structures, ideas from computer science and signal processing become useful. For example the fast Walsh-Hadamard transform - a type of discrete fourier transform turning the dyadic convolution (bitwise XOR convolution) into multiplication; and the Berlekamp-Massey theorem for discovering the shortest linear recurrence generating a sequence of elements of a field. In this direction, here are a few of my favourite problems:

- `https://projecteuler.net/problem=560` - Walsh-Hadamard Transform.
- `https://projecteuler.net/problem=644` - A brilliant problem which is a sort of continuous version of Rayles (see later on!)
- `https://projecteuler.net/problem=260` - A simple first example which can be sieved using the $\mathcal{P}$-position, $\mathcal{N}$-position argument mentioned earlier. There is however a deeper combinatorial theory lurking behind.
- `https://projecteuler.net/problem=459` - Solved using the field of nimbers. This is discussed later!

3 Misère impartial games

The simple theory of normal play impartial games is not enjoyed when the winning conditions are interchanged. In misère play, the player who makes the final move loses. This seemingly minor change renders the elegant Sprague-Grundy theory unable to help, with no similarly powerful analogue in misère play. This is in part because the simple identity $G + G = 0$, fundamental to the structure of the nim sum, no longer holds.

Notice that it is still easy to evaluate the outcome classes of various positions recursively as $\mathcal{P}$ and $\mathcal{N}$-positions interact in the same fashion as before. Whilst this technique is useful for initial investigations, it is unable to exploit the disjunctive sum structure, among other issues.

There are two main techniques for analysing misère impartial games: genus theory and the misère quotients. Genus theory attempts to recreate the Sprague-Grundy theory of normal play impartial games - a global theory which assigns to each position a *genus symbol*, containing generalised nim values describing how the position interacts with sums of heaps of misère nim of size 2. The theory works nicely for a subclass of games known as *tame*. Tame games behave like misère play nim - however almost all games feature *wild* positions, which put up a great deal of resistance against this approach.

Genus theory attempts to recreate the Sprague-Grundy theory too closely. It doesn't seem reasonable to classify all misère impartial games in a uniform way, for example there are more than $2^{2^{4171779}}$ inequivalent misère games 7 or fewer moves long yet only 8 such games for normal play (due to the Sprague-Grundy theorem) [Con]. It turns out that we can sacrifice the global applicability of the theory to retain its functionality. We follow [Sie].

3.1 Misère quotients

Let $o^-(G)$ denote the outcome class of the game G under misère play. For a set of game positions $\mathcal{A}$, we say that two positions A and B are equivalent when their outcome classes interact with the all other positions in the same manner.

$$A \cong_{\mathcal{A}} B \iff o^-(G+X) = o^-(H+X) \quad \forall X \in \mathcal{A}$$

We require that a sum of positions in $\mathcal{A}$ remains in $\mathcal{A}$, so we replace $\mathcal{A}$ by its closure $cl(\mathcal{A})$ by including all sums of all positions and their options. It is easy to verify that $\cong_{\mathcal{A}}$ is an equivalence relation on positions of $\mathcal{A}$.

Lemma 6 *Let $A \cong_{\mathcal{A}} B$ and $K \in \mathcal{A}$, then $A + K \cong_{\mathcal{A}} B + K$*

Proof. By associativity of the disjunctive sum of games, for all $X \in \mathcal{A}$

$$o^-((A+K)+X) = o^-(A+(K+X)) = o^-(B+(K+X)) = o^-((B+K)+X)$$

Shows the claim. So $\cong_{\mathcal{A}}$ is a congruence.

Definition 7 A monoid *is a set S equipped with a binary operation $\odot$ subject to the following conditions.*

1. There is an identity element $1 \in S$ such that $1 \odot x = x$ for all $x \in S$,
2. $x, y \in S \Rightarrow x \odot y \in S$,
3. $\odot$ is associative.

Consider the quotient $\mathcal{Q} = \mathcal{A}/_{\cong_{\mathcal{A}}}$. Lemma 6 ensures that the operation $\odot$ given by $[A] \odot [B] = [A+B]$ is well defined. (2) holds by closure of $\mathcal{A}$. The identity is the equivalence class containing the empty position and associativity holds by associativity of +: $[A] \odot ([B] \odot [C]) = [A] \odot [B+C] = [A+(B+C)] = [(A+B)+C] = [A+B] \odot [C] = ([A] \odot [B]) \odot [C]$. Similarly, note that $\odot$ is commutative. Therefore $\mathcal{Q}$ is a commutative monoid, but not necessarily a group. We cannot guarantee that each element has an inverse, unlike in normal play where $G + G = 0$. In the case where $\mathcal{A}$ is taken to be the set of all normal play impartial games, this construction will produce an infinite direct sum of copies of $\mathbf{Z}/2$. The equivalence classes are represented by nim heaps and $\odot$ is the nim sum.

Define the *pretending function* to be the quotient map $\Phi : \mathcal{A} \to \mathcal{Q}, G \mapsto [G]$. and $\mathcal{P} \subset \mathcal{Q}$ to represent the $\mathcal{P}$-positions of $\mathcal{A}$. $\mathcal{P} = \{[x] : x \in \mathcal{P}\}$. Note that the outcome class of a position is well defined: $A \cong_{\mathcal{A}} B \Rightarrow o^-(A+0) = o^-(B+0)$.

Definition 8 *The* Misère quotient $\mathcal{Q}(\mathcal{A})$ *of* $\mathcal{A}$ *is the pair* $(\mathcal{Q}, \mathcal{P})$.

The appeal of this construction is that if a game is written as a disjunctive sum of generators $(X_1, \ldots, X_n)$, then we can determine the outcome class of a position $X_{k_1} + \ldots + X_{k_i}$ by checking whether $\Phi(X_{k_1} + \ldots + X_{k_i}) = \Phi(X_{k_1}) \odot \ldots \odot \Phi(X_{k_i}) \in \mathcal{P}$. This is identical to $(*)$ in the previous section for normal play impartial games. Indeed, with this, we provide a way to classify misère positions in the same way as the Sprague-Grundy theory so successfully does with normal play positions. The only catch is we have to determine the structure of $\mathcal{Q}$, which changes for each set of nonequivalent games. The determination of the misère quotient is not an easy task and much of the literature on misère quotients concerns their algorithmic computation. There is also a comprehensive tool for doing just this called *MisereSolver* [Pla].

Note that nowhere in the construction of the misère quotient, was the misere win condition used. This construction can be used equally well to analyse normal play impartial games, indeed the old theory is implied by the misère quotient, often called the *indistinguishability quotient* for general games. We will use this unified construction to prove a theorem about a certain class of impartial games for both normal and misère rules simultaneously.

4 Periodicity of Φ for octal games

4.1 Octal games

We introduce a natural class of impartial games played with of heaps of stones called *taking and breaking games*. Each move removes stones from a single heap and then possibly splits that heap into two new heaps of sizes chosen by the player.

Definition 9 An *octal code* **.abc**$\cdots$ is a compact notation for representing taking and breaking games. The games are represented by an octal number in which the i^{th} digit's binary expansion features:

2^0 If a pile of i stones can be destroyed.

2^1 If a pile of $n > i$ stones can have i stones removed from it.

2^2 If a pile of $n > i + 1$ stones can have i stones removed from it, and then be split into two new heaps.

Example 10 *Rayles* is played with points in the plane. The players take turns drawing closed loops passing through either one or two points such that each loop does not self-intersect or intersect any other loop. We verify that rayles is the octal game **.77**. The connected components of the plane minus the loops are the heaps and each heap has the same structure regardless of the positions of its points. the digit **7** represents that within a connected component, any loop must partition the connected component into two disjoint regions in which one or both regions may contain zero points. There are two non-zero digits in the octal code because a loop must either pass through 1 or 2 points.

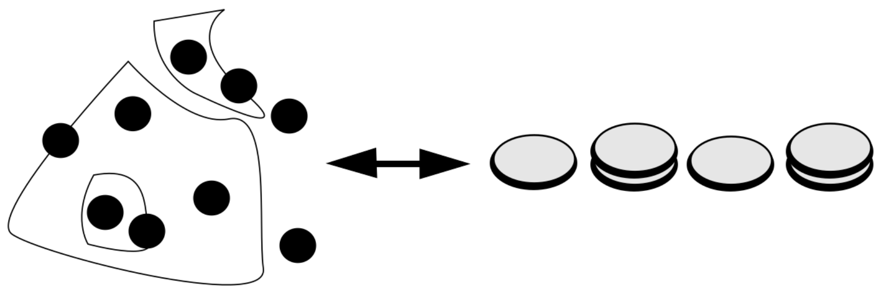

Figure 1: The correspondence between rayles and its octal game on piles of stones.

4.2 Periodictiy of octal games

Definition 11 For a game $(H_k)_k$ indexed by $\mathbf{N}$, the n^{th} *partial quotient* $\mathcal{Q}_n(\mathcal{A})$:= $\mathcal{Q}(\mathcal{A}_n)$, where $\mathcal{A}_n = cl\{H_0, \ldots, H_n\}$ is the closure of the first $n+1$ games.

We will make use of the following lemma which we will not prove.

Lemma 12 The pretending function of the options G' of G uniquely determines $\Phi(G)$. So if there is a some game $H \in \mathcal{A}$ such that $\Phi(\{H' : H'$ option of $H\}) = \Phi(\{G' : G'$ option of $G\})$ then $\Phi(H) = \Phi(G)$ and also $\mathcal{Q}(\mathcal{A} \cup \{G\}) \cong \mathcal{Q}(\mathcal{A})$.

We do however note that this is true for normal play impartial games: the pretending function identifies positions with nim heaps, which are determined uniquely by the equivalent nim heaps of their options by the *mex* operation, so the lemma is evident in normal play.

We modify approaches of [Elw] and [Sie] to give the following unified proof of a powerful periodicity theorem for both normal and impartial octal games.

Theorem 13 (Periodicity condition for octal games) Let $G = (G_n)_n$ be an octal game with last non-zero octal code digit in position k. Fix n_0 and p and set $M = 2n_0 + 2p + k$. Let $\Phi_M : \mathcal{A}_M \to \mathcal{Q}_M$ be the pretending function of the M^{th} partial quotient. Then if $\Phi_M(G_{n+p}) = \Phi_M(G_n)$ for all $n_0 \leq n < 2n_0+p+k$ then $\mathcal{Q}_M(G) \cong \mathcal{Q}(G)$ and there is periodicity of the pretending function with period p from this point on: $\Phi(G_{n+p}) = \Phi(G_p)$ for $n \geq n_0$.

Proof. A move in position G_n is of the form $G_n \to G_a+G_b$ with $n-k \le a+b < n$. Assume that the periodicity of Φ_M holds for $n_0 \le n < 2n_0 + p + k$. Now take $n \ge 2n_0+p+k$. Each move $G_{n+p} \to G_a+G_b$ satisfies $a+b \ge n+p-k \ge 2n_0+2p$ so we can select $a \ge n_0 + p$ without loss of generality. Now by assumption, $\Phi_M(G_{a-p}) = \Phi_M(G_a)$ and

$$\Phi_M(G_{a-p}+G_b) = \Phi_M(G_{a-p}) \odot \Phi_M(G_b) = \Phi_M(G_a) \odot \Phi_M(G_b) = \Phi_M(G_a+G_b)$$

If $(n+p)-(a+b)$ is a valid decrement of stones for one move, then so is $n-((a-p)+b)$. Therefore if $G_{n+p} \to G_a + G_b$ is a valid move, so is $G_n \to G_{a-p} + G_b$. So for each option of G_{n+p} there is some option of G_n with equal Φ_M value. By lemma 12, $\Phi_M(G_{n+p}) = \Phi_M(G_n)$ and the periodicity holds by induction. Similarly by lemma 12, for each $r \ge 1$, $\mathcal{Q}(\mathcal{A} \cup \{G_{M+1}, \ldots, G_{M+r}\}) \cong \mathcal{Q}(\mathcal{A})$. So again by induction $\mathcal{Q}(\mathcal{A}) \cong \mathcal{Q}(\mathcal{A}_M) = \mathcal{Q}_M(\mathcal{A})$. Finally Φ_M extendeds to Φ in the natural way and $\Phi(G_{n+p}) = \Phi(G_p)$ for $n \ge n_0$.

Modulo lemma 12, we have proved a criterion for delayed periodicity of the pretending function of octal games. In the normal play case, this implies periodicity of the nim values for octal games. Using this we can solve the game of rayles using a simple computer search to tabulate the nimbers until the periodicity conditions are met.

4.3 Solving rayles in normal play

By the Sprague-Grundy theorem, it suffices to solve the game for single heaps. The nim value of a position is the *mex* of the nim values of its options and we can compute these values recursively by simply trying every possible move: Let $\mathcal{G}(n)$ denote the nim value of a heap of size n, then

$$\mathcal{G}(n) = mex(\{\mathcal{G}(i) \oplus \mathcal{G}(n-i-2) | 0 \le i \le n-2\} \cup \{\mathcal{G}(i) \oplus \mathcal{G}(n-i-1) | 0 \le i \le n-1\})$$

with initial conditions $\mathcal{G}(0) = 0$, $\mathcal{G}(1) = 1$ and $\mathcal{G}(2) = 2$. A period $p = 12$ sequence emerges, starting for $n_0 = 71$. **.77** has two digits so $k = 2$. Since the periodicity holds up to $M = 2n_0 + 2p + k = 168$, this periodicity persists forever by theorem 13. Here's the nim values up to $n = 168$.

0	1	2	3	1	4	3	2	1	4	2	6
4	1	2	7	1	4	3	2	1	4	6	7
4	1	2	8	5	4	7	2	1	8	6	7
4	1	2	3	1	4	7	2	1	8	2	7
4	1	2	8	1	4	7	2	1	4	2	7
4	1	2	8	1	4	7	2	1	8	6	7
4	1	2	8	1	4	7	2	1	8	2	7
4	1	2	8	1	4	7	2	1	8	2	7
4	1	2	8	1	4	7	2	1	8	2	7
4	1	2	8	1	4	7	2	1	8	2	7
4	1	2	8	1	4	7	2	1	8	2	7
4	1	2	8	1	4	7	2	1	8	2	7
4	1	2	8	1	4	7	2	1	8	2	7
4	1	2	8	1	4	7	2	1	8	2	7

The misère analysis is similar and the *MisereSolver* software can be used to verify that misère **.77** also has period 12.

Many octal games have been shown to exhibit a delayed periodic behaviour however it is an open problem whether all octal games are eventually periodic. For example, the game **.106** has period $p = 328226140474$ with preperiod $n_0 = 465384263797$ and many other octal games are currently showing no signs of periodicity in their nim values [Fla].

5 Tartan games and the field of nimbers

In this section we use [Fer] and [Con] to develop theory in order to solve a problem from [Eul]. In this section we assume the normal play convention.

We have seen in section 2 that the nim sum endows **N** with the structure of an abelian group. We will now introduce a product which makes **N** into a field. This product is in some sense the simplest product which endows a field structure to **N** and in [Con], Conway constructs the product based on this assumption.

Coin turning games are played on a line of coins with one face black and the other white. The players take turns flipping coins in arrangements specified by the rules, such that the rightmost coin turned goes from black to white. For example, take the coin turning game known as *ruler* in which a legal move flips a contiguous strip of coins of any length such that the rightmost coin is black.

We will use the augmented structure of nimbers to solve the following class of games. *Tartan games* are the direct product of two coin turning games $G_1 \times G_2$, with with legal moves as the direct product of legal moves of G_1, G_2. Explicitly, flipping all coins in position (x_i, y_j) for $0 \leq i \leq a, 0 \leq j \leq b$ is a legal move iff flipping coins $x_0, \ldots, x_a$ is a legal move in G_1 and flipping coins $y_0, \ldots, y_b$ is legal in G_2. Each move must also have top right coin showing black. This ensures that the game has finite length and so the Sprague-Grundy theory applies.

Definition 14 The *nim product* is defined recursively as

$$a \otimes b := mex\{(a \otimes b') \oplus (a' \otimes b) \oplus (a' \otimes b') | 0 \leq a' < a, 0 \leq b' < b\}$$

with $0 \otimes x = x \otimes 0 = 0$.

Theorem 15 $(\mathbf{N}, \oplus, \otimes)$ *is a field with one as* 1 *and zero as* 0.

We proceed without proving theorem 15, as the proof is not particularly illuminating. We will instead apply the theorem to solve an interesting game!

5.1 The tartan theorem

One dimensional coin turning games In a coin turning game, let M be the set of nim values of all possible moves from the start position $S_k = WW \dots WB$ with k white coins. Let v_k denote the nim value of S_k, then the following characterises the game:

$$v_k = mex\{v_k \oplus x | x \in M\} \quad (**)$$

Remark 17 The nim value of a game with black coins in positions X, and white elsewhere has nim value $\bigoplus_{i \in X} \mathcal{G}(i)$, where position i is the position with all white coins except one black coin in position i.

This can be seen by considering an equivalent game (in normal play). Instead of flipping coins, a move now flips the rightmost coin and adds a new game, played disjunctively, with black counters corresponding to the coins which would have been flipped in the coin turning game, not including the rightmost coin. This game has equivalent nim values to the original game, as instead of flipping a counter from black to white, the disjunctive sum of games with black counters in the same position cancels as $x \oplus x = 0$. Note that this structure is therefore not enjoyed in misère play. Generalising this, the nim value of a collection of coins in a tartan game $G_1 \times G_2$ is the nim sum of the nim values of the individual black coins in the position.

We now design a proof of a remarkable theorem which allows the nim value of the direct product of coin turning games to be computed analogously to how the nim sum works for the disjunctive sum of games $(*)$. We will require the integral domain structure of the nimbers.

Remark that the positions of a tartan game often resemble the layered striped patchwork pattern of tartan (maybe slightly visible in Fig. 2).

Theorem 18 (The tartan theorem) *Denote the Sprague-Grundy function of G_i by $\mathcal{G}_i$, then the nimbers of the game $G_1 \times G_2$ are $\mathcal{G}(x,y) = \mathcal{G}_1(x) \otimes \mathcal{G}_2(y)$.*

Proof. Suppose that the tartan theorem holds for $0 \leq x' \leq x, 0 \leq y' \leq y$ and $(x',y') \neq (x,y)$. Denote the coins that are flipped by a move in G_i by X_i. Then using the mex rule and remark 17, where we iterate over all moves X_1, X_2 of G_1, G_2:

$$\mathcal{G}(x,y) = mex\{ \bigoplus_{x' \in X_1, y' \in X_2} (\mathcal{G}_1(x') \otimes \mathcal{G}_2(y')) \oplus (\mathcal{G}_1(x) \otimes \mathcal{G}_2(y))\}$$

Using distributivity of the nim sum and product,

$$\mathcal{G}(x,y) = mex\{(\bigoplus_{x' \in X_1} \mathcal{G}_1(x')) \otimes (\bigoplus_{y' \in X_2} \mathcal{G}_2(y')) \oplus (\mathcal{G}_1(x) \otimes \mathcal{G}_2(y))\}$$

$$= mex\{(x_1 \otimes x_2) \oplus (\mathcal{G}_1(x) \otimes \mathcal{G}_2(y)) | x_1 \in M_1, x_2 \in M_2\} = mex(T).$$

Using the characterisation of one dimensional coin turning games $(**)$ $\mathcal{G}_1(x) = mex\{\mathcal{G}_i(x) \oplus x_i | x_i \in M_i\}$, we see that $x_1, x_2 \neq 0$, so $x_1 \otimes x_2 \neq 0$ and $(x_1 \otimes x_2) \oplus (\mathcal{G}_1(x) \otimes \mathcal{G}_2(y)) \neq \mathcal{G}_1(x) \otimes \mathcal{G}_2(y)$. Since $(**)$, there is an $x_i = \mathcal{G}_i(x) \oplus t_i$ for each $0 < t_i < \mathcal{G}_i(x)$, as each $t_i < \mathcal{G}_i(x)$ must be in $\{\mathcal{G}_i(x) \oplus x_i | x_i \in M_i\}$.

Therefore, for each $0 \leq t_1 < \mathcal{G}_1(x)$ and $0 \leq t_2 < \mathcal{G}_2(y)$, $(\mathcal{G}_1(x) \otimes \mathcal{G}_2(y)) \oplus (\mathcal{G}_1(x) \otimes t_1) \oplus (\mathcal{G}_2(y) \otimes t_2) \in T$. Using distributivity, we see that $(\mathcal{G}_1(x) \otimes t_2) \oplus (\mathcal{G}_2(x) \otimes t_1) \oplus (t_1 \otimes t_2) \in T$. Therefore $\mathcal{G}(x,y) \geq mex\{(\mathcal{G}_1(x) \otimes t_2) \oplus (\mathcal{G}_2(x) \otimes t_1) \oplus (t_1 \otimes t_2) | 0 \leq t_1 < \mathcal{G}_1(x), 0 \leq t_2 < \mathcal{G}_2(y)\} = \mathcal{G}_1(x) \otimes \mathcal{G}_2(y)$ Since $\mathcal{G}_1(x) \otimes \mathcal{G}_2(y)$ is excluded from T, we conclude that $\mathcal{G}(x,y) = \mathcal{G}_1(x) \otimes \mathcal{G}_2(y)$.

Now the tartan theorem holds for the entire non-negative integer lattice by induction.

5.2 A 2D coin flipping game

Consider the following tartan game played on a square of black coins of width n. A legal move is to flip a rectangle of coins with width a square number and height a triangular number. The rectangle must as usual have a black top right corner. The last player able to move wins. We ask for an efficient algorithm to count the number of first moves that win the game. This question is posed in [Eul] - we now provide a solution.

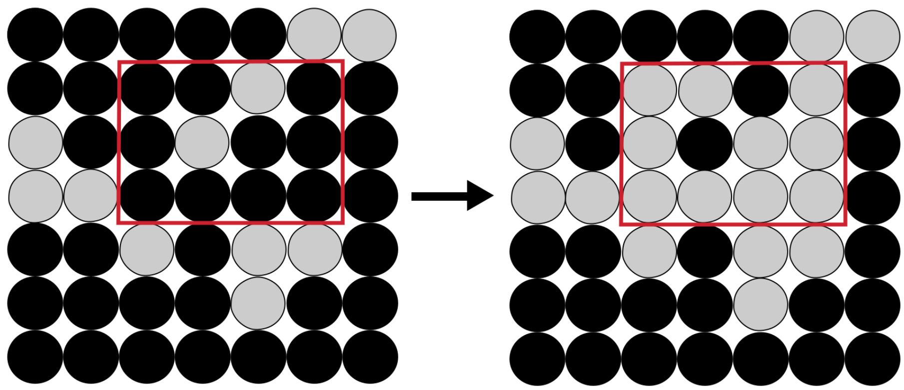

Figure 2: An example move from our 2D coin flipping game.

By the tartan theorem, the nim values of this game can be deduced from the nim values of two one dimensional coin turning games using the nim product. So to begin, we solve these one dimensional games.

Notice that since each move flips a contiguous strip of coins, moves from $WWWWWWWB$ look like $WWWBBBW$. It suffices to only compute the value of $C_k := \mathcal{G}(1) \oplus \ldots \oplus \mathcal{G}(k)$, as the nimber of a contiguous strip from a to b is $C_b \oplus C_{a-1}$. Therefore, C_{k+1} can be computed as $C_{k+1} = mex\{C_k \oplus C_{k-i} | i \in I\}$, where I is the set of permissible move lengths, a nice improvement from $\mathcal{G}(k+1) = mex\{\mathcal{G}(k) \oplus \ldots \oplus \mathcal{G}(k-i) | i \in I\}$.

Nim values of rectangles From the nim values g_1, g_2 of the one dimensional games, the nim value of a rectangle $R = (W = [a, b], H = [c, d])$ can be computed as $\mathcal{G}(R) = \bigoplus_{i \in W, j \in H} g_1(i) \otimes g_2(j)$. Using the distributivity property of the field of nimbers, this expands as $(\bigoplus_{i \in W} g_1(i)) \otimes (\bigoplus_{j \in H} g_2(j))$. Since the nim sums range over contiguous strips, we can compute these using the C_k of the one dimensional games as $\mathcal{G}(R) = (C^1_b \oplus C^1_{a-1}) \otimes (C^2_d \oplus C^2_{c-1})$. Since we will precompute the C_k, the nim value of a move can be deduced using only three operations, $\oplus$ being instantaneous and $\otimes$ being efficient as we will now show.

Efficient nim multiplication The following facts are given in [Con]. Define $F_n = 2^{2^n}$.

1. $F_n \otimes x = xF_n$ for $x < F_n$
2. $F_n \otimes F_n = \frac{3}{2} F_n$

Notice that all powers of two 2^n can be written as a product of the F_i in a unique way by expressing the exponent n in binary. So we compute the nim sum recursively: let $2^m, 2^n$ be the largest power of two smaller than x, y respectively, then

$x \otimes y = (2^m \oplus x') \otimes (2^n \oplus y') = (2^m \otimes 2^n) \oplus (2^m \otimes y') \oplus (x' \otimes 2^n) \oplus (x' \otimes y')$. It remains to compute $2^m \otimes 2^n$.

It is easy to show that $a \otimes b \leq ab$. Now using property (2),
$(\bigotimes_{i \in I \subset \{1,2,\ldots n\}} F_i) \otimes (\bigotimes_{j \in J \subset \{1,2,\ldots n\}} F_j) \leq \frac{3}{2} \prod_{i=0}^{n} F_i = \frac{3}{2} 2^{2^{n+1}-1} < F_{n+1}$.

So if 2^n and 2^m don't share a common highest factor of F_i, the factor can be extracted. Assume the highest factor is F_k of 2^n, then $2^n \otimes 2^m = F_k((2^n/F_k) \otimes 2^m)$. This process can be repeated until the arguments share a common factor, in which case we proceed recursively by extracting the common factor using (2) as $\frac{3}{2} F_k \otimes ((2^n/F_k) \otimes (2^m/F_k))$.

Counting winning positions Winning first moves have nimber 0. The nim value of a first move is simply the nim value of the entire board N, nim summed with the nimber of the first move $M = \mathcal{G}(width) \otimes \mathcal{G}(height)$. To search for $M = N$, we first iterate over all contiguous strips in the associated one dimensional games to form a frequency table of their nimbers F_1, F_2. Then, iterating over the values $y \in V_1$ of F_1, we solve $y \otimes x = N$. Note that it suffices to solve $y' \otimes x = 1$ and solutions always exist as the nimbers form a field. The number of winning first moves of the two dimensional coin turning game is then $\Sigma_{x \in V_1} F_1(x) F_2(x^{-1})$. All that remains is to find an efficient way to divide by nimbers.

In [Con], Conway's construction of the field of nimbers reveals that the inverse of a nimber n is not larger than the smallest F_k larger than n. So it suffices to compute the inverse of nimbers by brute force, caching the result, provided the nimbers do not get too large. Which for this problem, we find the nimbers do not exceed 512 when the initial board measures 10^6 by 10^6. So computing the inverse of nimbers poses no computational threat and we have an efficient solution.

Exercise: thanks for reading! Consider the coin turning game in which each move turns a rectangle of coins with any width but height a multiple of three. Each rectangle must have a black coin in the top right corner. The start position is given in figure 2. You may play either first or second, which do you choose?

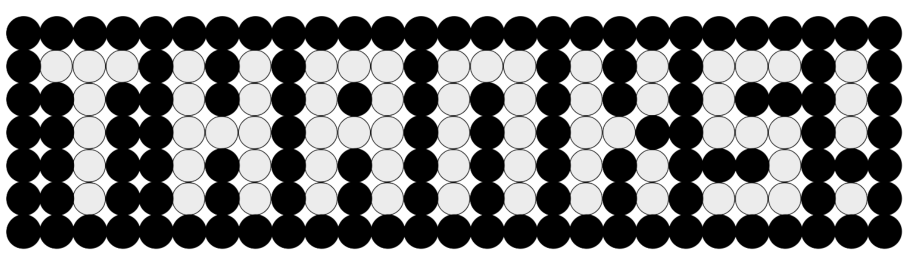

Figure 3: Do you play first or second?

Solution. Using the tartan theorem, the entire board filled with black coins can be shown to have nim value 43. Similarly, the nim value of the text is also 43. So the start position has nim value 0 and is a second player win.

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