From Spinors to Twistors: Eliminating Redundancies in Scattering Amplitudes

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Submitted: April 17, 2018

The scattering amplitude of an interaction may be used to determine its cross-section and thereby the probability of its occurrence in a particle collider. Thus, a reliable computation of the scattering amplitude may easily be compared to experiment, thereby providing a useful tool for evaluating the successes and failures of the Standard Model. Ultimately this implies that highly accurate computation of scattering amplitudes could be particularly crucial in developing a more fundamental theory of particle physics. Scattering amplitudes, however, can be computationally expensive to calculate, especially for greater multiplicities and at higher orders. This is a result of multiple redundancies present in the theory, which may be removed by a number of techniques that have been developed for this purpose. In this paper, an outline of some of these techniques is given and examples are provided so as to demonstrate their efficacy.

Contents

1. INTRODUCTION

A. Quantum Chromodynamics

In quantum field theories, scattering amplitudes are traditionally calculated from a set of Feynman rules which themselves may be extracted from the Lagrangian. This paper concentrates on the interactions between gluons which may occur in the deep inelastic scattering of hadrons, present in colliders such as the LHC and the RHIC. The entire Standard Model Lagrangian therefore need not be considered, and only the part of it that describes the strong interaction (otherwise known as the colour interaction) between quarks and gluons is relevant to this paper. This part of the Lagrangian is known as the quantum chromodynamics (QCD) sector and is given by the expression,

$$\mathcal{L}_{QCD} = \sum_{k=\{u,d,c,s,t,b\}} \bar{q}_k(i\not{D} - m_k)q_k - \frac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu}, \tag{1}$$

where the *gauge covariant derivative*, D_μ is given by,

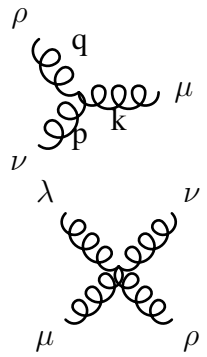
$$D_\mu \equiv \partial_\mu + ig_s T^a \mathcal{A}^a_\mu, \tag{2}$$

and the *gluon field strength tensor*, $G^{\mu\nu}_a$ is given by,

$$G^{\mu\nu}_a \equiv \partial_\mu \mathcal{A}^a_\nu - \partial_\nu \mathcal{A}^a_\mu + g f^{abc} \mathcal{A}^b_\mu \mathcal{A}^c_\nu. \tag{3}$$

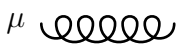
The sum in the original expression is over the six species of quark, the fields and masses for which are represented by q_k and m_k respectively. g_s is the strong coupling constant. T^a are the Gell-Mann matrices representing the colour symmetry $SU(3)$ group, $\mathcal{A}^a_\mu(x)$ the gluon gauge fields and f^{abc} the structure constants of $SU(3)$, where the a, b, c indices refer to the eight independent colour states of gluons.

The reason that this Lagrangian was derived to predict the relevant experimental results and continues to do so with great success, stems from certain observed symmetries. The complete derivation of the Lagrangian will not be described in this paper, however a good starting point for the inquisitive reader can be found in Peskin and Schroeder's book entitled '*An Introduction to Quantum Field Theory*' [1] or alternatively in '*Quantum Chromodynamics*' by Greiner, Schramm and Stein [2]. The derivation of the Feynman rules from the QCD Lagrangian is also omitted, in favour of stating the relevant *colour-ordered* rules for gluon interactions *ipse dixit*,



$$= \frac{i}{\sqrt{2}}(\eta_{\nu\rho}(p-q)_\mu + \eta_{\rho\mu}(q-k)_\nu + \eta_{\mu\nu}(k-p)_\rho)$$

$$= i\eta_{\nu\rho}\eta_{\nu\lambda} - \frac{i}{2}(\eta_{\mu\nu}\eta_{\rho\lambda} + \eta_{\mu\lambda}\eta_{\nu\rho}) \quad , \tag{4}$$



$$= i\frac{\eta_{\mu\nu}}{p^2},$$

n	3	4	5	6	7	8	9	10
Feynman diagrams	1	4	25	220	2485	34300	559405	10525900
Ordered diagrams	1	3	10	38	154	654	2871	12925
MHV	1	1	1	1	1	1	1	1
BCFW NMHV	-	-	-	3	6	10	15	21
BCFW N^2MHV	-	-	-	-	-	20	50	105
BCFW N^3MHV	-	-	-	-	-	-	-	175

FIG. 1: The number of graphs appearing in tree-level n-gluon scattering using different approaches. The number of terms in BCFW are chosen for the most complicated alternating helicity case. [3]

where $\eta_{\mu\nu}$ is the metric tensor[1] and p, q and k depict 4-momenta, i.e. vectors in Minkowski space.[2] The meaning and significance of *colour-ordering* shall be explained in Section 2B.

To apply the rules, it is necessary only to perform the product of each rule corresponding to each vertex or internal propagator within the Feynman diagram of the relevant process. It is important also to impose total momentum conservation at each vertex and to include polarisation vectors for each external gluon leg, the form of which shall be detailed in Section 2A.

B. Outline of Redundancies

It is apparent from Figure 1 that, when calculating scattering amplitudes using QCD (or indeed any quantum field theory) via the Feynman rules an extraordinary number of diagrams must be considered for even quite modest numbers of external gluons at tree-level. For example, amplitudes involving ten external gluons require the summation of over ten million diagrams and yet, a special type of amplitude known as an MHV (Maximally-Helicity-Violating) amplitude need consider only one.[3] The reason for such an incongruous surplus of diagrams comes from the fact that many are related by gauge invariance and are therefore not required to be computed individually. Such redundancy arises both from the $SU(3)$ colour symmetry of QCD as well as from the admission of off-shell quantities which are unphysical and therefore must cancel when the final, gauge-invariant scattering amplitude is computed. Thus, finding a way to procedurally compute amplitudes in terms of on-shell contributions is most beneficial to reducing the arduousness of scattering amplitude computation.

In the forthcoming section, the Weyl spinor formalism will be introduced, in which quarks are approximated to be massless. This trivialises the on-shell condition,

$$p^2 = m^2, \tag{5}$$

[1] In this paper the mostly-plus convention is used: $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$.

[2] Minkowski space is the space formed by combining 3D Euclidean space and time into a 4D manifold so as to preserve the spacetime interval between any two events upon a change of inertial frame. The contravariant four-momentum of a particle with relativistic energy E and three-momentum $\boldsymbol{p} = (p_x, p_y, p_z)$ is given by the expression, $p = (\frac{E}{c}, p_x, p_y, p_z)$ where c is the speed of light.

[3] MHV amplitudes are amplitudes for which $n-2$ of the n external gluons share the same helicity. NMHV amplitudes are those for which $n-3$ share the same helicity and so on. Helicity is the projection of the spin-operator onto the direction of motion of a particle [4].

where p^2 is equal to the square of the four-momentum of the particle $p^\mu p_\mu$ and m is equal to its mass. Thus, the on-shell condition becomes,

$$p^2 = 0. \tag{6}$$

The formalism also allows amplitudes to be expressed in a manifestly Lorentz-invariant fashion. This results from the Lorentz-covariance of the Dirac Lagrangian. It is then possible to apply a number of techniques aimed at removing redundancies by exploiting the symmetries of the theory.

One such technique, mentioned previously is that of *colour-ordering*, in which the colour redundancy of the the $SU(3)$ group is removed and only *gauge-invariant* partial amplitudes need be considered. Further detail on colour-ordering is given in Section 2B.

A further useful technique, explained in Section 2C, will seek to remove off-shell redundancy. Ordinarily, Feynman rules require calculating every partial amplitude starting from off-shell expressions. The external particles are then taken to be on-shell only at the very end of the calculation. This new technique, *BCFW recursion* requires the computation of only on-shell amplitudes (i.e. those that do not vanish when (6) is satisfied). This reduces the number of DoFs (degrees of freedom) to align with the physical number of DoFs, since non-virtual particles are required to be on-shell in order to be considered physical.

Finally, it will be demonstrated in Section 3 how a new formalism, *twistor* formalism, can incorporate all of the aforementioned restrictions, whilst enabling a further reduction of DoFs, related to the conservation of the total momentum of all external gluons. This symmetry is only made manifest in twistor formalism as a result of the implicit total momentum conservation, which is not present in the spinor formalism.

Thus, the employment of these techniques allows for a number of redundancies present in the initial formulation to be duly eliminated, enabling an increase in the efficiency of scattering amplitude computation, as well as a deeper understanding of the theory of gluon interactions.

2. SPINORS

A. Formalism

A conveniently compact notation for expressing and condensing amplitudes derived from Feynman diagrams is given by spinor-helicity formalism. Spinors are solutions to the Dirac equation. It is therefore no surprise that they show up in the QCD Feynman rules, given that the QCD Lagrangian contains within it the Dirac Lagrangian, which itself contains information about the dynamics of fermions such as quarks.

The formalism relies upon the approximation of massless fermions or *Weyl* fermions, which holds in the high-energy limit and allows the solutions of the Dirac equation to be separated from four-component Dirac spinors into two two-component objects,

$$u_+(k_i) = v_-(k_i) \equiv \begin{pmatrix} 0 \\ |k_i\rangle^{\dot{a}} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ |i\rangle^{\dot{a}} \end{pmatrix}, \quad u_-(k_i) = v_+(k_i) \equiv \begin{pmatrix} |k_i]_a \\ 0 \end{pmatrix} \equiv \begin{pmatrix} |i]_a \\ 0 \end{pmatrix}, \tag{7}$$

where k_i refers to the ith out of a set of n momenta.

Each of these objects, known as Weyl spinors, have frame-independent helicity; one of the objects $|i\rangle^{\dot{a}}$ has positive helicity and is known as the right-handed Weyl spinor and the other $|i]_a$ has negative helicity and is known as the left-handed Weyl spinor.[4] The dotted and undotted indices refer to the fact that the two types of spinors form different representations of the $SU(2)$ group in which they lie ($(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ respectively).

In fact, both types of Weyl spinor combined form an irreducible representation of the proper Lorentz group,

$$SU(2) \otimes SU(2) \to SO(1,3), \tag{8}$$

from which a more general (massive) solution to the Dirac equation can always be built [5].

Anti-spinors, which represent outgoing anti-fermions (as opposed to the outgoing fermions represented by spinors) are represented by taking the Dirac conjugate of their corresponding spinor. This results in left-handed and right-handed Weyl anti-spinors being represented by $[i|^a$ and $\langle i|_{\dot{a}}$ respectively. For real-valued momenta, spinors and anti-spinors are related by,

$$[i|^a = (|i\rangle^{\dot{a}})^*, \;\; \langle i|_{\dot{a}} = (|i]_a)^*, \tag{9}$$

such that a right-handed anti-spinor may be interpreted as a left-handed conjugate spinor and similarly for a left-handed anti-spinor.

Thus, the spinor products may be written,

$$\begin{aligned} \langle ij\rangle &\equiv \langle i|_{\dot{a}}|j\rangle^{\dot{a}} = \overline{u_-(k_i)}u_+(k_j), \\ [ij] &\equiv [i|^a|j]_a = \overline{u_+(k_i)}u_-(k_j). \end{aligned} \tag{10}$$

Spinor products of the same helicity vanish as a result of the helicity projection [6], thus no shorthand is required for their expression.

Expressing the momentum of a particle in terms of its corresponding spinors is simple,

$$[i|\sigma^\mu|i\rangle = \langle i|\bar{\sigma}^\mu|i] = 2k_i^\mu. \tag{11}$$

This is known as *Gordon's identity*.

A number of other spinor identities can be shown which allow manipulation of spinorial scattering amplitudes in order to reach more compact forms; many can be found in Dixon's 1996 paper [7]. It now stands to introduce the spinor representation of the polarisation vector for a massless gauge boson of definite helicity ± 1. The representation can be shown to have the desired properties and is presented *ipse dixit*,

$$\epsilon_\mu^+(k,q) = \frac{\langle q|\bar{\sigma}_\mu|k]}{\sqrt{2}\langle qk\rangle}, \quad \epsilon_\mu^-(k,q) = -\frac{[q|\sigma_\mu|k\rangle}{\sqrt{2}[qk]}, \tag{12}$$

where k is the momentum of the boson and q is an arbitrary *reference momentum* reflecting the on-shell gauge freedom, which may be chosen for convenience. Certain choices will often allow multiple terms and diagrams to vanish, simplifying the computation considerably. Some useful relevant identities are given in the aforementioned paper. σ^μ and $\bar{\sigma}^\mu$ are defined as follows,

$$\sigma^\mu = (1, \sigma_1, \sigma_2, \sigma_3), \;\; \bar{\sigma}^\mu = (1, -\sigma_1, -\sigma_2, -\sigma_3), \tag{13}$$

[4] Note that in the massless limit helicity and chirality are equivalent.

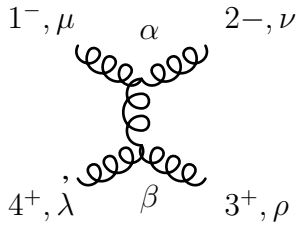
where σ_i are the Pauli matrices[5].

An additional point to note is that the helicities assigned to particles are dependent upon whether they are incoming or outgoing. In this paper the convention is to label particles with a helicity as if it is outgoing; if instead it is wished for it to be expressed as incoming, the true helicity is the reverse of that given in the label.

B. An Example

Now that the rules and notation have been characterised, it serves to provide an illustrative example. In this paper only *tree amplitudes*, i.e. amplitudes which do not contain loops, are discussed. The discussion may also be extended to higher order amplitudes, in which loops may be present, however tree amplitudes possess certain properties which allow powerful reductive techniques to be applied as shall be demonstrated later in this paper.

Consider the extremely simple case of a scattering process involving four gluons to leading order, for which the scattering amplitude is depicted, $A_4^{tree}(1^-, 2^-, 3^+, 4^+)$. Such a process is discussed in Dixon's aforementioned paper, where the author has made the astute choice of gluon reference momenta, $q_1 = q_2 = k_4$, $q_3 = q_4 = k_1$, such that a number of the relevant diagrams vanish. Following the approach of Dixon's paper, it can be seen that the only non-vanishing diagram that remains is,



(14)

where both the momentum-helicity labels and the Lorentz indices have been explicitly added for clarity.

Using the Feynman rules outlined on the previous page, the invariant scattering amplitude is therefore given by,[6]

$$\begin{aligned} A_4^{Tree}(1^-, 2^-, 3^+, 4^+) = \frac{i\eta_{\alpha\beta}}{s_{12}} \cdot \frac{i}{\sqrt{2}}(\eta_{\mu\nu}(k_1 - k_2)_\alpha + \eta_{\nu\alpha}(k_2 - \sqrt{s_{12}})_\mu + \eta_{\alpha\mu}(\sqrt{s_{12}} - k_1)_\nu) \\ \cdot \frac{i}{\sqrt{2}}(\eta_{\rho\lambda}(k_3 - k_4)_\beta + \eta_{\lambda\beta}(k_4 - \sqrt{s_{34}})_\rho + \eta_{\beta\rho}(\sqrt{s_{34}} - k_3)_\lambda) \\ \cdot \epsilon_\mu^-(1, q_1) \cdot \epsilon_\nu^-(2, q_2) \cdot \epsilon_\rho^+(3, q_3) \cdot \epsilon_\lambda^+(4, q_4) \end{aligned} \tag{15}$$

where the factors $s_{ij} \equiv (k_i + k_j)^2 = 2k_i \cdot k_j$ come about as a result of squaring the propagator momenta, which are related to the external momenta through total momentum conservation.

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[5] $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

[6] Note that to keep things simple, the explicit portrayal of covariance and contravariance in the tensor indices has been neglected, in favour of "dotting" the rules together in an intuitive manner. The reader is assured that this is consistent with a more notationally rigorous approach.

The last equality comes from the fact that the external momenta are on-shell and massless. For example, the momentum of the internal gluon, k_g may be given by $k_g + k_1 + k_2 = 0$ which implies that ${k_g}^2 = (k_1 + k_2)^2 = s_{12}$. It may similarly be given in terms of the total momentum conservation at the opposite vertex.

Contracting the indices may be done computationally or (somewhat tediously) by hand. Doing so, and making use of total momentum conservation at each vertex, the following expression may be obtained,

$$\begin{aligned} A_4^{Tree}(1^-, 2^-, 3^+, 4^+) = &\left(\frac{i}{\sqrt{2}}\right)^2 \left(\frac{-i}{s_{12}}\right) \\ &\times \left[\epsilon_1^- \cdot \epsilon_2^- (k_1 - k_2)^\mu + (\epsilon_2^-)^\mu \epsilon_1^- \cdot (2k_2 + k_1) + (\epsilon_1^-)^\mu \epsilon_2^- \cdot (-2k_1 - k_2)\right] \\ &\times \left[\epsilon_3^+ \cdot \epsilon_4^+ (k_3 - k_4)_\mu + (\epsilon_4^+)_\mu \epsilon_3^+ \cdot (2k_4 + k_3) + (\epsilon_3^+)^\mu \epsilon_4^+ \cdot (-2k_3 - k_4)\right]. \end{aligned} \tag{16}$$

At this point, the choice of reference momenta allows all but the $\epsilon_2^- \cdot \epsilon_3^+$ contractions to vanish. Thus, after the final contraction is performed,

$$A_4^{Tree}(1^-, 2^-, 3^+, 4^+) = -\frac{2i}{s_{12}} (\epsilon_2^- \cdot \epsilon_3^+)(\epsilon_1^- \cdot k_2)(\epsilon_4^+ \cdot k_3). \tag{17}$$

Substituting in the expressions for the polarisation vectors and making use of the identities.

$$\begin{aligned} |i]\langle i| &= \sigma_\mu k_i^\mu \\ |i\rangle[i| &= \bar{\sigma}_\mu k_i^\mu, \end{aligned} \tag{18}$$

the expression may be written in a spinorial form,

$$A_4^{Tree}(1^-, 2^-, 3^+, 4^+) = -\frac{2i}{s_{12}} \left(-\frac{2}{2}\frac{[43]\langle 12\rangle}{[42]\langle 13\rangle}\right) \left(-\frac{[42]\langle 21\rangle}{\sqrt{2}[41]}\right) \left(+\frac{\langle 13\rangle[34]}{\sqrt{2}\langle 14\rangle}\right) \tag{19}$$

which, after some trivial algebra, yields,

$$A_4^{Tree}(1^-, 2^-, 3^+, 4^+) = -\frac{\langle 12\rangle[34]^2}{[12]\langle 14\rangle[14]}. \tag{20}$$

This may be reduced further through judicial use of identities to yield a compact and simple final expression,

$$A_4^{Tree}(1^-, 2^-, 3^+, 4^+) = i\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 41\rangle}, \tag{21}$$

or

$$A_4^{Tree}(1^-, 2^-, 3^+, 4^+) = i\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{22}$$

In practice, it may be necessary to calculate the amplitudes for each helicity configuration and sum over the results so as to obtain the total scattering amplitude of a four-gluon interaction.

C. Colour-Ordering

In this example, it may have been apparent how convenient spinor notation is for expressing scattering amplitudes in a compact manner, albeit after some extensive algebra. It is also the

case, however, that finding the most compact expression becomes more and more difficult for higher-order diagrams and that the number of diagrams that need be interpreted increases exponentially with multiplicity, thereby making computation increasingly onerous. This has already been partially addressed by applying colour-ordering, the significance of which shall now be explained.

With regards to colour-ordering, the reduction in diagrams occurs as a result of a general property of the colour structure of tree amplitudes which allows a process called *colour-decomposition* to occur. The original Feynman rules (not given in this paper) contain structure constants f^{abc} for each gluon three-vertex and contracted pairs of structure constants $f^{abe}f^{cde}$ for each gluon four-vertex (see Section 1 for more details). These are defined in terms of the Gell-Mann matrices T^a, i.e. the group generators of $SU(3)$, via $\left[T^a, T^b\right] = i\sqrt{2}f^{abc}T^c$. Using this definition, and the fact that the $SU(3)$ generators must form the complete set of traceless hermitian 3×3 matrices, it can be shown that any tree amplitude may be reduced to a sum of single trace terms (see [8] for details),

$$\mathcal{A}_n^{tree}(k_i, \lambda_i, a_i) = g_s^{n-2} \sum_{\sigma \in S_n/Z_n} Tr(T^{a\sigma(1)} \dots T^{a\sigma(n)}) A_n^{tree}(\sigma(1^{\lambda_1}), \dots, \sigma(n^{\lambda_n})), \tag{23}$$

where k_i and λ_i are the gluon momenta and helicities respectively and g_s is the strong coupling constant. S_n is the set of all permutations of n objects whereas, Z_n is the subset of cyclic permutations.

Thus, it has become evident that only the *partial amplitudes*, which are of a particular cyclic ordering, need to be calculated explicitly, before summing over the rest of the permutations. Crucially, the partial amplitudes are simpler to compute by virtue of their colour-ordering.

D. BCFW Recursion

Whilst colour-ordering limits the number of diagrams that need to be computed, it is still the case that diagrams corresponding to large multiplicities produce extremely convoluted expressions as a result of their many vertices. BCFW recursion, named after its original proposers Britto, Cachazo, Feng and Witten, makes use of two known three-gluon amplitudes,

$$\begin{aligned} A_3(1^-, 2^-, 3^+) &= \frac{\langle 12 \rangle^3}{\langle 23 \rangle \langle 31 \rangle}, \\ A_3(1^+, 2^+, 3^-) &= -\frac{[12]^3}{[23][31]}, \end{aligned} \tag{24}$$

in order to recursively determine the tree-level amplitude of any gluon scattering process regardless of its complexity, thereby bypassing the need to explicate long, convoluted expressions as well as reducing further the number of graphs to compute. In fact, BCFW recursion may also be extended to include massive particles [9] as well as diagrams containing loops [10], although that is beyond the scope of this report. The all-plus or all-minus three-gluon amplitudes are not needed, since it may be shown that all such amplitudes, regardless of multiplicity, must vanish. A simple way to see this is by choosing all reference momenta of the polarisation vectors to be equivalent and to note that $\epsilon_i^+(q) \cdot \epsilon_j^+(q) = \epsilon_i^-(q) \cdot \epsilon_j^-(q) = 0$.

1. Derivation

The derivation of the BCFW recursion relation (as outlined coherently in Truijen's paper [11], which this paper closely follows) relies on the analytical properties of tree amplitudes. In order to investigate these properties, the application of a momentum shift or *deformation* is performed. This deformation involves two (and only two) of the external momenta being analytically continued into the complex plane. In general, such a shift for two arbitrary external gluons with momenta, k_i and k_j takes the form,

$$\begin{aligned} &k_i \to k_i(z) = \hat{k}_i = k_i + zq, \\ &k_j \to k_i(z) = \hat{k}_j = k_j - zq, \\ &k_i \cdot q = 0 = k_j \cdot q, \;\; q^2 = 0, \;\; z \in \mathbb{C}. \end{aligned} \tag{25}$$

A specific deformation oft used in literature is known as the $[i|j\rangle$-deformation,

$$\begin{aligned} &|i\rangle = |\hat{i}\rangle \;\; = |i\rangle\,, \quad |i] = |\hat{i}] \;\; = |i] + z|j], \\ &|j] = |\hat{j}] \;\; = |j], \quad |j\rangle = |\hat{j}\rangle \;\; = |j\rangle - z\,|i\rangle\,, \\ &z \in \mathbb{C}. \end{aligned} \tag{26}$$

It can be seen that the arbitrarily complicated amplitude, $A(z)$ after such a deformation is necessarily meromorphic as a result of the structure of the Feynman rules, a property which turns out to be extremely useful. This is because the only non-analytic parts are the isolated propagator poles which occur when the propagator momenta are on-shell for some value of z.

In order to see how this meromorphic property may prove useful, the following integral is investigated,

$$\mathcal{B} = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz \frac{A(z)}{z}, \tag{27}$$

where $\mathcal{C}$ is given by,

$$\mathcal{C} = \lim_{R\to\infty} \mathcal{C}_R, \;\; \mathcal{C}_R = \{z \in \mathbb{C} | z = Re^{i\theta}, 0 \leq \theta \leq 2\pi\}. \tag{28}$$

Using the meromorphic property, it is thus possible to apply Cauchy's residue theorem such that the calculation of the integral reduces to,

$$\mathcal{B} = \sum_{z_a} \text{Res}(\frac{A(z)}{z}, z_a), \tag{29}$$

where z_a are the values of z corresponding to poles and their residues in the function $\frac{A(z)}{z}$.

The most obvious pole occurs at $z = 0$, which has a residue $A(0) = A$, whereas the rest come from the propagator poles within $A(z)$. Thus, it is possible to write,

$$\mathcal{B} = A + \sum_{z_a \neq 0} \text{Res}(\frac{A(z)}{z}, z_a). \tag{30}$$

If the vanishing condition,

$$\lim_{|z|\to\infty} A(z) = 0, \tag{31}$$

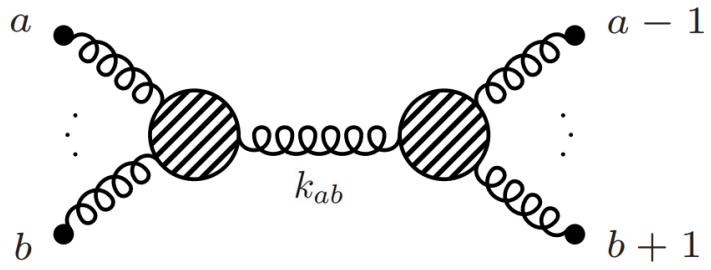


FIG. 2: An arbitrarily complicated tree amplitude in which a single propagator is isolated.

holds, then it follows that the integral given by $\mathcal{B}$ vanishes and the original amplitude, A may be determined if the poles of $A(z)$ and their corresponding residues are known. Indeed, it can be shown that for the $[i|j\rangle$-deformation, the vanishing condition holds as long as the helicities are chosen such that $(h_i, h_j) = (-,+), (+,+)$ or $(-,-)$ [12]. In the case of $(h_i, h_j) = (+,-)$, the interchange of i and j ensures that this can always be achieved.

In order to determine the poles of $A(z)$, it is instructive to recall that at tree-level they arise when the propagator momenta are on-shell.

Due to the cyclical arrangement of colour-ordering as well as total momentum conservation at the vertices, each propagator momentum may be expressed as a sum of the momenta of neighbouring external gluons attached to either one of the sides (see Figure 2). Thus, a propagator momentum takes the general form, $k_{ab}(z) = k_a(z) + k_{a+1}(z) + \ldots + k_b(z)$, where $a, b \in \{1, \ldots, n\}$.

Using the convenient notation, $\hat{k}_{ab} = k_{ab}(z), k_{ab} = k_{ab}(0)$, it is clear that the poles arise when,

$$\hat{k}_{ab}^2 = 0. \tag{32}$$

Recalling (25), this can only occur when i and j are on separate sides of the propagator such that only one is contained within the set $\{a, \ldots, b\}$, otherwise $\hat{k}_{ab}$ becomes independent of z and there is therefore no pole. Assuming without loss of generality that i is contained within the set, such that $\hat{k}_{ab} = k_{ab} + zq$, the condition given by (32) implies that,

$$z = z_{ab} = -\frac{k_{ab}^2}{2q \cdot k_{ab}}. \tag{33}$$

In particular, for the $[i|j\rangle$-deformation,

$$z = z_{ab} = -\frac{k_{ab}^2}{[j|k_{ab}|i\rangle}. \tag{34}$$

Thus the pole for each propagator has been determined exclusively in terms of its momentum and the value of q associated with the shift.

Turning now to the residue associated with each pole, it is necessary to understand the behaviour of $A(z)$ when $z \to z_{ab}$ which equates to when $\hat{k}_{ab} \to 0$. In fact, from the factorisation properties

detailed in *Dixon* [13], it follows that $A(z)$ factorises into two on-shell sub-amplitudes in the aforementioned limit. Thus, the residue at the pole z_{ab} may be found according to,

$$\text{Res}(\frac{A(z)}{z}, z_{ab}) = -\sum_{\lambda} A(a,\ldots,b,-\hat{k}_{ab}^{\lambda})|_{z_{ab}} \frac{1}{k_{ab}^2} A(\hat{k}_{ab}^{-\lambda}, b+1,\ldots,a-1)|_{z_{ab}}, \tag{35}$$

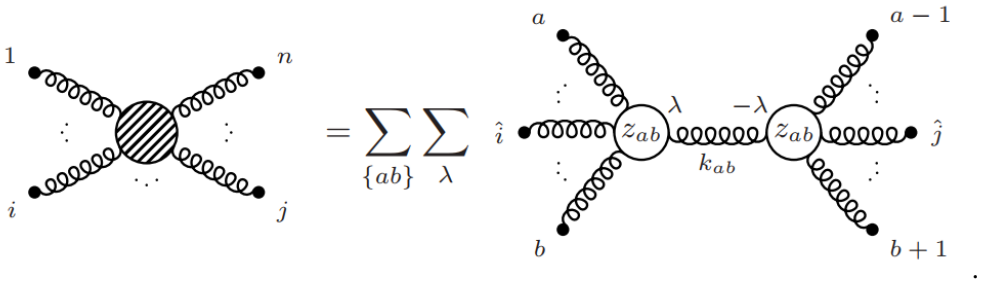
where the sum is performed over both positive and negative helicities and the evaluation subscript z_{ab} refers to the fact that $\hat{k}_i$ and $\hat{k}_j$ must be evaluated at the value of z corresponding to the pole, whereas the squared propagator term is evaluated at $z = 0$.

Crucially, each side contains fewer than n gluons, and it therefore immediately follows from the previous discussion that the amplitude may be expressed in terms of the following recursive relation,

$$\begin{aligned} A &= \sum_{\{ab\}\in\mathcal{O}} \sum_{\lambda} A(a,\ldots,b,-\hat{k}_{ab}^{\lambda})|_{z_{ab}} \frac{1}{k_{ab}^2} A(\hat{k}_{ab}^{-\lambda}, b+1,\ldots,a-1)|_{z_{ab}}, \\ \mathcal{O} &= \{\{ab\}|i \in \{ab\} \wedge j \notin \{ab\}\} \end{aligned} \tag{36}$$

where $\{ab\} = \{a,\ldots,b\}$. This is none other than the BCFW recursion relation.

Alternatively, the relation may be expressed diagrammatically [14] as,



(37)

2. *Vanishing Sub-Amplitudes*

At the top of this section, it was stated that all all-plus and all-minus helicity amplitudes must vanish,

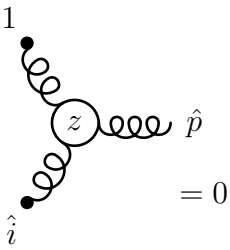
$$A_n(1^{\pm},\ldots,n^{\pm}) = 0 \;\; \forall \;\; n. \tag{38}$$

It may also be shown similarly, by choosing reference momenta $q_1 = k_n, q_2 = \ldots = q_n = k_1$, that all all-but-one-plus and all-but-one-minus helicity amplitudes must vanish for more than 3 external gluons,

$$A_n(1^{\mp}, 2^{\pm},\ldots,n^{\pm}) = 0 \;\; \forall \; n > 3. \tag{39}$$

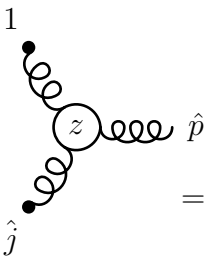
BCFW recursion with an $[i|j\rangle$-deformation ensures that two more sub-amplitudes vanish in the special cases of their containing the $\hat{i}$ and $\hat{j}$ gluons. Specifically, it can be shown [15] that,

$$A_i(+,+,-) = \quad = 0, \tag{40}$$



where the helicities may be assigned to each leg in any combination. Similarly, it may also be shown that,

$$A_j(-,-,+) = \quad = 0. \tag{41}$$



In demonstrating that a number of sub-amplitudes vanish, it can easily be seen that BCFW diagrams consisting solely of these particular sub-amplitudes vanish, without having to compute them. Thus, the number of contributing diagrams to an amplitude for any tree-level process is greatly reduced (at least for $n > 3$).

In addition to this, BCFW recursion involves computing exclusively *on-shell* sub-amplitudes, thereby removing any redundancy caused by off-shell propagator poles which may cancel between diagrams. This also has the advantage of eliminating the risk of large numerical errors occurring from such poles [16].

3. An Example

An example of the use of BCFW recursion can be seen in the derivation of the Parke-Taylor formula,

$$A_n^{MHV}(1^+,\ldots,x^-,\ldots,y^-,\ldots,n^+) = \frac{\langle xy\rangle^4}{\langle 12\rangle\ldots\langle n-1n\rangle\langle n1\rangle}, \tag{42}$$

which holds for mostly-plus MHV amplitudes [17, 18]. A similar formula also holds for mostly-minus MHV amplitudes [19] (otherwise known as $\overline{\text{MHV}}$ amplitudes),

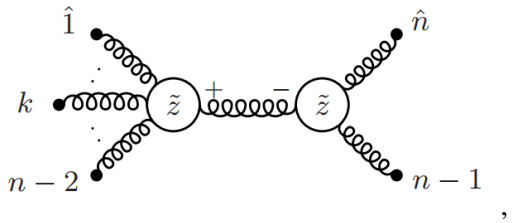
$$A_n^{MHV}(1^-,\ldots,x^+,\ldots,y^+,\ldots,n^-) = \frac{[xy]^4}{[12]\ldots[n-1n][n1]}, \tag{43}$$

and may also be proven similarly with BCFW recursion, although such a proof is omitted in this paper.

Comparison with (22) and (24) shows that the Parke-Taylor formula holds in the $n = 3$ and $n = 4$ cases. Thus, a proof by induction may be attempted to extend the proof to $n > 4$.

Arranging the amplitude as $A_n(1^-,\ldots,k^-,\ldots,n^+)$ and applying a $|1n\rangle$-deformation it can be easily seen from the vanishing sub-amplitudes that only a single BCFW diagram contributes, namely,

$$A_n(1^-,2^+,\cdots,k^-,\cdots,n^+) = \tag{44}$$



,

where $\tilde{z} = z_{1,n-2} = \frac{\langle n-1,n\rangle}{\langle n-1,1\rangle}$.

Using induction for the left sub-amplitude and the known expression for the right sub-amplitude, the amplitude may be expressed as,

$$\begin{aligned} A_n(1^-,\ldots,k^-,\ldots,n^+) &= \frac{\langle \hat{1}k\rangle^4}{\langle \hat{1}2\rangle\ldots\langle n-2,\hat{k}_{n-1,n}\rangle\langle\hat{k}_{n-1,n}1\rangle} \\ &\times \frac{1}{\langle n,n-1\rangle[n-1,n]}\frac{-[n-1,\hat{n}]^3}{[\hat{n},\hat{k}_{1,n-2}][\hat{k}_{1,n-2},n-1]} \\ &= \frac{\langle 1k\rangle^4}{\langle 12\rangle\ldots\langle n-2,\hat{k}_{n-1,n}\rangle\langle\hat{k}_{n-1,n}1\rangle} \\ &\times \frac{1}{\langle n,n-1\rangle[n-1,n]}\frac{-[n-1,n]^3}{[n,\hat{k}_{n-1,n}][\hat{k}_{n-1,n},n-1]}, \end{aligned} \tag{45}$$

where $k_{a,b} = k_a + \ldots + k_b$.

It then follows from,

$$\begin{aligned} \langle n-2,\hat{k}_{n-1,n}\rangle[n,\hat{k}_{n-1,n}] &= -\langle n-2,n-1\rangle[n-1,n], \\ \langle\hat{k}_{n-1,n}1\rangle[\hat{k}_{n-1,n},n-1] &= -\langle 1n\rangle[n,n-1], \end{aligned} \tag{46}$$

that this is equivalent to the Parke-Taylor formula for n gluons,

$$A_n(1^-,\ldots,k^-,\ldots,n^+) = \frac{\langle 1k\rangle^4}{\langle 12\rangle\ldots\langle n-1,n\rangle\langle n1\rangle}, \tag{47}$$

as required.

3. TWISTORS

A. Motivation

Whilst BCFW recursion has proven to be an invaluable technique, it can often lead to expressions involving a convoluted mess of spinor products, meaning a considerable degree of assiduousness is necessary in order to represent them in their most compact form. Indeed it is not always immediately obvious how two expressions are supposed to be equivalent. Take, for example, the algebra required to reduce (45) to (47). In practice, this means that an impractical amount of computation would be required to simulate interactions to the accuracy required by the current second run of the LHC using spinors [20]. Further to this, in the spinorial partial amplitudes of non-MHV amplitudes, there remain singularities, or *spurious poles* which are not present in the total amplitude [21].

The fact that these superfluous expressions exist within the BCFW spinor formalism reflects a hidden symmetry that has not been made manifest. A relatively recent formalism known as *momentum twistor* formalism has been suggested in hope of rectifying this issue. Momentum twistors were first proposed by Hodges [22] as an application of Roger Penrose's twistor theory

in the context of scattering amplitudes, and a number of authors have since had some success in applying the theory to modern high-multiplicity loop calculations [23–25]. Elsewhere in this paper, momentum twistors shall simply be referred to as twistors.

The advantage of twistors as opposed to spinors is that a hidden symmetry, *dual conformal symmetry*, can be naturally expressed in twistor coordinates and thereby made manifest [26].

B. Dual Conformal Symmetry

Whilst spinors transform linearly under Lorentz transformations, thereby resulting in manifestly Lorentz-invariant amplitudes, they do not transform linearly under a general Poincaré transformation. The Lorentz group is, in fact, a subgroup of the Poincaré group containing only the isometries[7] that leave the origin unchanged. This means that spinors do not transform linearly under translations in momentum-space and thus expressions containing spinor products contain added redundancy in the form of non-implicit total momentum conservation. Hence, for n external momenta, the condition,

$$\sum_{i=1}^{n} k_i = 0, \tag{48}$$

must be explicitly imposed to produce a fully compact amplitude.

In spinor formalism, therefore, total momentum conservation must generally be imposed in a rather *ad hoc* manner. The partial amplitudes are determined from the Feynman rules, and then a delta function $\delta^4(\sum_{i=1}^{n} k_i)$ is applied to each so as to simplify the expression. An example of this can be seen in (45) where between the first and second lines, the denominator of the rightmost fraction has been reexpressed by virtue of total momentum conservation, which amounts to multiplication by the aforementioned delta function.

Twistors, by contrast, make use of new *dual space*[8] coordinates which enable them to produce manifestly dual-conformally invariant amplitudes, formed from objects with implicit total momentum conservation without the need for the *ad hoc* application of delta functions.

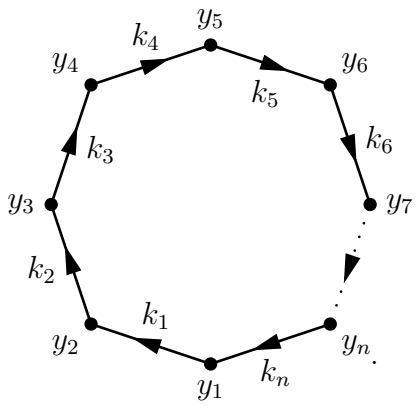
In fact, dual-conformal symmetry pertains to more than just total momentum conservation and reflects additional symmetries only present in massless particles. The conformal group is an extension of the Poincaré group which includes special conformal transformations and dilations. This results in a total of 15 symmetries as opposed to the 10 in the Poincaré group and the 6 in the Lorentz group. Thus, amplitudes expressed in terms of twistors are invariant under the full range of relevant transformations, meaning that no identities are required in order to simplify them. It might therefore be expected that scattering amplitudes expressed in terms of twistor coordinates lend themselves to more efficient numerical computation.

[7] An isometry is a linear transformation that preserves length.

[8] The dual space corresponding to a vector space (in this case momentum space) consists of all linear functionals on the vector space together with pointwise addition and scalar multiplication.

C. Formalism

To begin to understand these new coordinates, it is useful to consider a geometrical interpretation of total momentum conservation:



(49)

Laying each of the n external momentum vectors end to end forms a closed loop if total momentum is conserved. The *dual-space* coordinates y_i^μ may then be introduced to represent the points at which each pair of successive vectors meet, such that,

$$k_i^{\dot{a}a} = y_i^{\dot{a}a} - y_{i+1}^{\dot{a}a}. \tag{50}$$

Thus, the condition of momentum conservation may be expressed as $y_{n+1} = y_1$ or equivalently as $y_n = y_0$. Clearly, k_i must be invariant under a general translation in dual-space ($y_i \to y_i + c \ \forall \ i \in \{1, \ldots, n\}$, where c is a constant four-vector). Indeed, since colour-ordered partial amplitudes depend upon sums of consecutive k_i, it can be shown that they are invariant under any conformal transformation of dual-space, not just translations. Thus, expressing partial amplitudes in terms of dual-space coordinates ensures that they become manifestly dual-conformally invariant [27].

As solutions of the Weyl equations, the spinors must satisfy,

$$\langle i|\sigma_\mu k_i^\mu = 0, \quad [i|\bar{\sigma}_\mu k_i^\mu = 0, \tag{51}$$

from which it follows from (50) that $\langle i|_{\dot{a}}(\sigma \cdot y_i)^{\dot{a}a} = \langle i|_{\dot{a}}(\sigma \cdot y_{i+1})^{\dot{a}a}$ and similarly for $[i|^a$. Thus, for each dual-space coordinate, the following *incidence relations*,

$$\begin{aligned} [\mu_i|^a &= \langle i|_{\dot{a}}(\sigma \cdot y_i)^{\dot{a}a} &&= \langle i|_{\dot{a}}(\sigma \cdot y_{i+1})^{\dot{a}a}, \\ \langle \mu_i|_{\dot{a}} &= [i|^a(\bar{\sigma} \cdot y_i)_{a\dot{a}} &&= [i|^a(\bar{\sigma} \cdot y_{i+1})_{a\dot{a}}, \end{aligned} \tag{52}$$

hold, where $[\mu_i|^a$ and $\langle \mu_i|_{\dot{a}}$ are newly defined variables.

From this it can be seen that the line in y-space determined by any two points, y_i and y_{i+1}, may be defined in terms of a point in *Z-space*, $Z_i^I \equiv (|i\rangle^{\dot{a}}, [\mu_i|^a)$, where $I = (\dot{a}, a)$ is an $SU(2,2)$ index. These new four-component variables, Z_i^I are the objects referred to as twistors, which allow scattering amplitudes containing solely right-hand Weyl spinors to be reexpressed to ensure the manifestation of dual-conformal invariance.

To reexpress the left-handed Weyl spinors in a similar fashion, an object called the *dual-twistor* may also be defined, conveniently in terms of twistors,

$$W_i^A \equiv (\langle \mu_i|, |i]) \equiv \frac{\epsilon^{ABCD} Z_{(i-1)B} Z_{iC} Z_{(i+1)D}}{\langle i-1, i\rangle \langle i, i+1\rangle}, \tag{53}$$

ensuring that all left-handed Weyl spinors defined via the dual-twistor automatically satisfy total momentum conservation.

Thus, scattering amplitudes may be expressed in terms of components of twistors and dual-twistors, instead of directly as spinor products, ensuring that the required invariance is manifest. In order to ensure that the resulting amplitudes are in their most compact form, it is not necessary to apply a number of identities on a case by case basis as was the case for spinorial expressions, but merely to produce a canonical form of the twistors that can be applied to any number of colour-ordered expressions, which amounts to choosing a special frame to evaluate the kinematics.

D. Canonisation

In order to fix the required parameters, the n twistors involved in an interaction are first expressed in the form of a $4 \times n$ matrix,

$$\begin{pmatrix} |1\rangle & |2\rangle & \dots & |n\rangle \\ |\mu_1] & |\mu_2] & \dots & |\mu_n] \end{pmatrix}. \tag{54}$$

Such a matrix has $4n$ independent entries but must be transformed so as to contain only the $3n-10$ independent parameters associated with the physical number of DoFs of the system. The figure $3n-10$ may be understood as follows. Firstly, for n particles there are n four-momenta and therefore $4n$ potential DoFs. The trivialisation of the on-shell condition enabled by the massless approximation in the high-energy limit, described earlier in this paper, reduces this number to $3n$. The added constraint of total four-momentum conservation results in a further reduction by four DoFs. The final six reductions may be achieved by fixing the frame of the kinematics through a number of Lorentz transformations followed by a final rescaling of momenta. For example, the system may be rotated and boosted such that the first two momenta are fixed in opposite directions along the z-axis. This amounts to fixing three parameters. Then, the system may be boosted along the z-axis such that the magnitudes of the first two momenta are made equivalent, resulting in the fixing of the fourth parameter. A rotation may then be performed about the z-axis so as to bring the third momentum into the x-z plane, fixing the fifth parameter. The final parameter may then be fixed by rescaling the momenta such that $s_{12} = 1$.

A particularly neat canonical form for the Z and W matrices has recently been devised by Badger, [28] which makes use of the spinor-space equivalent (as opposed to momentum-space) of this very procedure. Firstly, the spinors are rotated such that the first two spinors take the form $\begin{pmatrix} c \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ c' \end{pmatrix}$, where c and c' are non-zero values. All of the spinors are then rescaled by a process called *little group scaling*[9] such that the first and second spinors take the form $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

[9] The little group (the $ISO(2)$ group in the massless case [29]), is defined as the subgroup of Lorentz transformations which leave the 4-momentum of on-shell particles invariant, therefore not affecting the overall amplitude. Such transformations correspond to a special combination of Lorentz boosts and rotations known as Wigner rotations [30]. Little-group scaling is the representation of these transformations in spinor space which amounts to the following mapping $|p\rangle \to t|p\rangle$, $|p] \to \frac{1}{t}|p]$. From (11) it can be seen that this leaves the momentum unaffected.

and $\begin{pmatrix}0\\1\end{pmatrix}$ respectively, whilst the remaining spinors take the form $\begin{pmatrix}c_i\\1\end{pmatrix}$ where c_i varies from spinor to spinor.

From these transformed spinors, the $|\mu_i]$s may then be determined with some choice of y_0 such that the first two $|\mu_i]$s are null. The Z-matrix is thus constructed. The W-matrix may then be derived via (53).

The final canonical form of the Z-matrix is given by,

$$\begin{pmatrix} \frac{\Sigma_i}{s_{12}} \\ 1-\delta_{1i} \\ \frac{\langle 123i\rangle\langle 34\rangle[23]}{\langle 1234\rangle\langle 1i\rangle[12]} \\ \frac{-\langle 13\rangle\langle 124i\rangle+\langle 14\rangle\langle 123i\rangle}{\langle 1234\rangle\langle 1i\rangle} \end{pmatrix}, \tag{55}$$

where,

$$\Sigma_i = \begin{cases} 0 & \text{i=1} \\ \frac{\langle 13\rangle\langle 2i\rangle}{\langle 23\rangle\langle 1i\rangle} & \text{otherwise} \end{cases}, \tag{56}$$

and $\langle ijkl\rangle$ is a conformally invariant object known as the 4-bracket and defined by,

$$\langle ijkl\rangle = \epsilon^{ABCD} Z_{iA} Z_{jB} Z_{kC} Z_{lD}. \tag{57}$$

For example, for $n = 4$ gluons, Badger's Z matrix is given by,

$$\begin{pmatrix} 1 & 0 & \frac{1}{s_{12}} & \frac{1}{s_{12}}\frac{\langle 13\rangle\langle 24\rangle}{\langle 23\rangle\langle 14\rangle} \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & \frac{1}{s_{12}} & \frac{\Sigma_4}{s_{12}} \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \tag{58}$$

where Σ_4 and s_{12} are the two independent parameters.

The corresponding W matrix is,

$$\begin{pmatrix} 0 & 0 & -\frac{s_{12}^2}{-1+\Sigma_4} & 0 \\ 0 & 0 & 0 & \frac{s_{12}}{-1+\Sigma_4} \\ 1 & s_{12} & -s_{12} & 0 \\ -1 & 0 & -\frac{s_{12}}{-1+\Sigma_4} & \frac{s_{12}}{-1+\Sigma_4} \end{pmatrix}. \tag{59}$$

A quick check that this formulation is behaving as it ought to may be achieved by verifying the Schouten identity,

$$\langle ij\rangle\langle kl\rangle = \langle ik\rangle\langle jl\rangle + \langle il\rangle\langle kj\rangle. \tag{60}$$

For example, for $i = 1, j = 2, k = 3, l = 4$,

$$\begin{aligned} \text{LHS} &= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} \frac{1}{s_{12}} & \frac{\Sigma_4}{s_{12}} \\ 1 & 1 \end{vmatrix} &&= \frac{1-\Sigma_4}{s_{12}} \\ \text{RHS} &= \begin{vmatrix} 1 & \frac{1}{s_{12}} \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} 0 & \frac{\Sigma_4}{s_{12}} \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & \frac{\Sigma_4}{s_{12}} \\ 0 & 1 \end{vmatrix} \times \begin{vmatrix} \frac{1}{s_{12}} & 0 \\ 1 & 1 \end{vmatrix} &&= \frac{1-\Sigma_4}{s_{12}} \end{aligned}. \tag{61}$$

E. An Example

As an example of the increase in efficiency provided by twistor formalism, consider the following expression taken from part of the 7-gluon NMHV amplitude,

$$\begin{aligned}\frac{1}{2i}\big(&\langle 24\rangle\langle 12\rangle[12][24] + \langle 12\rangle[25][12]\langle 25\rangle + [24]\langle 12\rangle[13]\langle 34\rangle + \langle 35\rangle\langle 12\rangle[13][25] \\ &+ \langle 45\rangle\langle 12\rangle[45][12] + \langle 24\rangle[12]\langle 13\rangle[34] + [35][12]\langle 13\rangle\langle 25\rangle \\ &+ [34]\langle 13\rangle\langle 34\rangle[13] + \langle 35\rangle[35]\langle 13\rangle[13] + \langle 45\rangle[45]\langle 13\rangle[13]\big)\end{aligned}, \tag{62}$$

which may be simplified, after an exhausting amount of algebra, to the form,

$$-\frac{1}{2i}\langle 1|2+3|4+5|6+7|1], \tag{63}$$

where $\langle 1|2+3|4+5|6+7|1]$ gives eight terms of the form $\langle 1|i|j|k|1] = \langle 1i\rangle[ij]\langle jk\rangle[k1]$ with $i \in \{2,3\}, j \in \{4,5\}, k \in \{6,7\}$. The initial expression requires 217 operations to compute, whereas the simplified expression requires 175.

Substituting either expression into Badger's Z matrix gives a unique expression,

$$\frac{is_{12}^2 X_4(Y_5 - Y_6 - X_5Y_7 + X_6Y_7 - X_6Y_5Y_7 + X_5Y_6Y_7)}{2(1-\Sigma_4)(\Sigma_5 - \Sigma_6)}, \tag{64}$$

where X_i and Y_i refer to the 3rd and 4th row of the ith column of the Z matrix respectively. This expression requires only 59 operations to compute, implying that a significant increase of computational efficiency has been gained by using twistor formalism.

4. CONCLUSIONS

This paper has demonstrated the efficacy of some of the techniques developed to improve scattering amplitude computation. The three main techniques covered were those of colour decomposition, BCFW recursion and momentum twistor formalism. The first technique removes the redundancy in the field theory resulting from $SU(3)$ colour symmetry and it was shown how this reduces the number of partial amplitudes that need be computed. The second technique removes the off-shell redundancy from amplitude calculations, whilst allowing diagrams to be calculated recursively. It was explained how this reduces the number of contributing partial amplitudes significantly and the particularly impactful example of the use of BCFW recursion in proving the Parke-Taylor formula for MHV amplitudes was given. Finally, an overview of momentum twistor formalism was presented, in which the role of twistors in implicitly restricting the amplitudes according to total momentum conservation was demonstrated. In particular, it was shown how Badger's canonical form of the twistor matrix significantly reduces the number of operations required in the computation of an expression included in the 7-gluon NMHV tree amplitude. Of course, in this paper the focus has been on tree-amplitudes, however the application of these techniques to one-loop amplitudes has already been demonstrated and it is conjectured that they may prove useful up to all loop orders [31].

Acknowledgments

I should like to express my sincerest gratitude to Prof. Simon Badger for his freely and enthusiastically given support with regards to this project, as well as to my supervisor Dr. Daniel Maitre for providing guidance and timely consultation throughout. Thanks also goes to Giuseppe DeLaurentis for providing me with spinorial expressions and to Prof. Valentin (Valya) Khoze for indulging me with a fascinating discussion on the super conformal group. A final thanks goes to Prof. Tom McLeish for persistently cultivating my passion for theoretical physics throughout his time as my academic adviser.

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Appendix

The code used for the example in Section 3E reads as follows. Use is made of the S@M package developed for Mathematica, documentation for which may be found here: https://arxiv.org/pdf/0710.5559v2.pdf.

```
createCanonicalZ[numMomenta_] := Module[{n = numMomenta},
  res = ConstantArray[0, {4, n}];
  res[[1]][[1]] = 1;
  res[[2]][[1]] = 0;
  res[[3]][[1]] = 0;
  res[[4]][[1]] = 0;
  For[i = 2, i ≤ n, i++,
   res[[1]][[i]] = Σ[i] / s12;
   res[[2]][[i]] = 1;
   res[[3]][[i]] = X[i];
   res[[4]][[i]] = Y[i]
  ];
  res[[3]][[n]] = 1;
  res]
```

```
wFromZ[z_] := Module[{Z = z},
  zWidth = Dimensions[Z][[2]];
  res = ConstantArray[0, {4, zWidth}];
  For[i = 1, i ≤ zWidth, i++,
   For[j = 1, j ≤ 4, j++,
    If[i - 1 <= 0, iminus = zWidth, iminus = i - 1];
    If[i + 1 > zWidth, iplus = 1, iplus = i + 1];
    res[[j]][[i]] = Sum[LeviCivitaTensor[4][[j, beta, gamma, delta]] Z[[beta]][[iminus]] Z[[gamma]][[i]] Z[[delta]][[iplus]], {beta, 4}, {gamma, 4}, {delta, 4}] /
       ({z[[1]][[iminus]], z[[2]][[iminus]]}.epsilon.{z[[1]][[i]], z[[2]][[i]]}) / ({z[[1]][[i]], z[[2]][[i]]}.epsilon.{z[[1]][[iplus]], z[[2]][[iplus]]})
   ];
   For[j = 3, j ≤ 4, j++,
    res[[j]][[i]] = -res[[j]][[i]]
   ]
  ];
  res
 ]
```

```
horrible =
 (Spaa[2, 4] * Spaa[1, 2] * Spbb[1, 2] * Spbb[2, 4] + Spaa[1, 2] * Spbb[2, 5] * Spbb[1, 2] * Spaa[2, 5] + Spbb[2, 4] * Spaa[1, 2] * Spbb[1, 3] * Spaa[3, 4] +
    Spaa[3, 5] * Spaa[1, 2] * Spbb[1, 3] * Spbb[2, 5] + Spaa[4, 5] * Spaa[1, 2] * Spbb[4, 5] * Spbb[1, 2] + Spaa[2, 4] * Spbb[1, 2] * Spaa[1, 3] * Spbb[3, 4] +
    Spbb[3, 5] * Spbb[1, 2] * Spaa[1, 3] * Spaa[2, 5] + Spbb[3, 4] * Spaa[1, 3] * Spaa[3, 4] * Spbb[1, 3] + Spaa[3, 5] * Spbb[3, 5] * Spaa[1, 3] * Spbb[1, 3] +
    Spaa[4, 5] * Spbb[4, 5] * Spaa[1, 3] * Spbb[1, 3]) / (2 * I)
```

$$-\frac{1}{2} i \left(\langle 1|2\rangle\langle 2|4\rangle[2|1][4|2] + \langle 1|2\rangle\langle 3|4\rangle[3|1][4|2] + \langle 1|3\rangle\langle 2|4\rangle[2|1][4|3] + \langle 1|3\rangle\langle 3|4\rangle[3|1][4|3] + \langle 1|2\rangle\langle 2|5\rangle[2|1][5|2] + \langle 1|2\rangle\langle 3|5\rangle[3|1][5|2] + \langle 1|3\rangle\langle 2|5\rangle[2|1][5|3] + \langle 1|3\rangle\langle 3|5\rangle[3|1][5|3] + \langle 1|2\rangle\langle 4|5\rangle[2|1][5|4] + \langle 1|3\rangle\langle 4|5\rangle[3|1][5|4]\right)$$

```
nice = -Sum[Spaa[1, i] * Spbb[i, j] * Spaa[j, k] * Spbb[k, 1], {i, 2, 3}, {j, 4, 5}, {k, 6, 7}] / (2 * I)
```

$$-\frac{1}{2} i \left(\langle 1|2\rangle\langle 4|6\rangle[4|2][6|1] + \langle 1|3\rangle\langle 4|6\rangle[4|3][6|1] + \langle 1|2\rangle\langle 5|6\rangle[5|2][6|1] + \langle 1|3\rangle\langle 5|6\rangle[5|3][6|1] + \langle 1|2\rangle\langle 4|7\rangle[4|2][7|1] + \langle 1|3\rangle\langle 4|7\rangle[4|3][7|1] + \langle 1|2\rangle\langle 5|7\rangle[5|2][7|1] + \langle 1|3\rangle\langle 5|7\rangle[5|3][7|1]\right)$$

```
n = 7
```

```
7
```

```
canonicalZ = Factor[Chop[createCanonicalZ[n]]];
canonicalW = Factor[wFromZ[canonicalZ]];
MatrixForm[canonicalZ]
```

$$\begin{pmatrix} 1 & 0 & \frac{1}{s12} & \frac{\Xi[4]}{s12} & \frac{\Xi[5]}{s12} & \frac{\Xi[6]}{s12} & \frac{\Xi[7]}{s12} \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & X[4] & X[5] & X[6] & 1 \\ 0 & 0 & 1 & 1 & Y[5] & Y[6] & Y[7] \end{pmatrix}$$

```
toTwistor[expr_] := Module[{x = expr},
  res = x;
  res = res /. Spaa[Sp[i_], Sp[j_]] :> ((epsilon.{canonicalZ[[1]][[i]], canonicalZ[[2]][[i]]}).{canonicalZ[[1]][[j]], canonicalZ[[2]][[j]]});
  res = res /. Spbb[Sp[j_], Sp[i_]] :> -(epsilon.{canonicalW[[3]][[i]], canonicalW[[4]][[i]]}).{canonicalW[[3]][[j]], canonicalW[[4]][[j]]};
  res]
```

```
horribleTwistor = toTwistor[horrible] // Chop // Factor // Chop
```

$$\frac{i\, s12^2\, X[4]\, (Y[5] - Y[6] - X[5]\, Y[7] + X[6]\, Y[7] - X[6]\, Y[5]\, Y[7] + X[5]\, Y[6]\, Y[7])}{2\, (-1 + \Xi[4])\, (\Xi[5] - \Xi[6])}$$

```
niceTwistor = toTwistor[nice] // Chop // Factor // Chop
```

$$\frac{i\, s12^2\, X[4]\, (Y[5] - Y[6] - X[5]\, Y[7] + X[6]\, Y[7] - X[6]\, Y[5]\, Y[7] + X[5]\, Y[6]\, Y[7])}{2\, (-1 + \Xi[4])\, (\Xi[5] - \Xi[6])}$$

```
niceTwistor === horribleTwistor
```

```
True
```

```
LeafCount[horrible]
LeafCount[nice]
LeafCount[niceTwistor]
```

```
217
```

```
175
```

```
59
```